STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

ZW 1968 - 012

A "set-theoretic" characterization of the compactness operator

by

G.A. Jensen

August 1968.

The Mathematical Centre at Amsterdam, founded the 11th of February, 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications, and is sponsored by the Netherlands Government through the Netherlands Organization for Pure Research (Z.W.O.) and the Central National Council for Applied Scientific Research in the Netherlands (T.N.O.), by the Municipality of Amsterdam and by several industries.

Introduction

The purpose of this note is to give a set-theoretical characterization of the compactness operator on a given set (c.f. problem 1, [1]). More precisely, given a set X and an operator $\sigma: 2^{2^X} \to 2^{2^X}$, necessary and sufficient conditions are obtained in order that σ be the compactness operator $\rho: 2^{2^X} \to 2^{2^X}$. Since our characterization uses a form of f.i.p., it is not the characterization that the authors of [1] desire. However, since it uses a weaker form of f.i.p. than is used in the definition of ρ, it is a step in the proper direction.

1. Definitions and basic properties.

For the sake of completeness, we include the definitions that are used and a list of properties of ρ that are needed for our characterization. We refer the reader to [1] and [2] for a more extensive study of ρ. X will be a fixed set throughout.

Definitions: 1. If $\mathcal{S} \in 2^{2^X}$, we let $\gamma\mathcal{S}$ denote all arbitrary intersections of finite unions of members of $\mathcal{S}$. Thus, $\gamma: 2^{2^X} \to 2^{2^X}$ is an operator.

2. The compactness operator $\rho: 2^{2^X} \to 2^{2^X}$ is defined by the following: for each $\mathcal{S} \in 2^{2^X}$,

$$\rho\mathcal{S} = \{ Y \subset X \mid \emptyset \neq \mathcal{T} \subset \mathcal{S} \text{ and } \mathcal{T} \cup \{Y\} \text{ has f.i.p.} \Rightarrow Y \cap (\cap \mathcal{T}) \neq \emptyset \}$$

3. If $\mathcal{S}, \mathcal{T} \in 2^{2^X}$, then

$$\mathcal{S} \wedge \mathcal{T} = \{ S \cap T \mid S \in \mathcal{S} \text{ and } T \in \mathcal{T} \}.$$

4. If $\lambda, \sigma: 2^{2^X} \to 2^{2^X}$, then $\lambda \leq \sigma$ if and only if $\lambda\mathcal{S} \subset \sigma\mathcal{S}$ for each $\mathcal{S} \in 2^{2^X}$.

5. Multiplication of operators is just the usual composition of functions.

PROPOSITION 1. The operators ρ and γ satisfy the following properties: for each $\mathcal{S} \in 2^{2^X}$,

1. $\rho\gamma = \rho$
2. $\gamma\mathcal{S} \wedge \rho\mathcal{S} \subset \rho\mathcal{S}$.
3. If $\emptyset \neq \mathcal{T} \subset \gamma\mathcal{S} \cap \rho\mathcal{S}$ and $\mathcal{T}$ has f.i.p., then $\cap\mathcal{T} \neq \emptyset$.
4. For each $A \subset X$, $\{ Y \mid Y \cap A \in \rho\mathcal{S} \} \subset \rho(\{A\} \wedge \mathcal{S})$.

Proof. Properties 1, 2 and 3 are proved in [1] and are relations (3), (5) and (4), respectively, of that paper.

To see the last property, suppose that $A \subset X$ and let $Y \subset X$ with $Y \cap A \in \rho \mathcal{S}$. Let $\emptyset \neq \mathcal{T} \subset \{A\} \wedge \mathcal{S}$ with $\mathcal{T} \cup \{Y\}$ having f.i.p.. It follows that $\mathcal{T} = \{A\} \wedge \mathcal{S}_1$ for some $\mathcal{S}_1 \subset \mathcal{S}$, $\mathcal{S}_1 \cup \{Y \cap A\}$ has f.i.p., and $\mathcal{S}_1 \neq \emptyset$. Since $Y \cap A \in \rho \mathcal{S}$, then $(Y \cap A) \cap (\bigcap \mathcal{S}_1) \neq \emptyset$ and so $Y \cap (\bigcap \mathcal{T}) \neq \emptyset$. Hence $Y \in \rho(\{A\} \wedge \mathcal{S})$.

2. Characterization of ρ.

LEMMA 1. Let $\sigma : 2^{2^X} \to 2^{2^X}$ be an operator which satisfies the following properties: for each $\mathcal{S} \in 2^{2^X}$,

(1) $\sigma\gamma = \sigma$

(2) $\gamma \mathcal{S} \wedge \sigma \mathcal{S} \subset \sigma \mathcal{S}$.

(3) If $\emptyset \neq \mathcal{T} \subset \gamma \mathcal{S} \cap \sigma \mathcal{S}$ and $\mathcal{T}$ has f.i.p., then $\bigcap \mathcal{T} \neq \emptyset$.

(4) For each $A \subset X$, $\{Y \mid Y \cap A \in \sigma \mathcal{S}\} \subset \sigma(\{A\} \wedge \mathcal{S})$.

Then $\sigma \leq \rho$.

Proof. Let $\mathcal{S} \in 2^{2^X}$ with $\gamma \mathcal{S} = \mathcal{S}$. Let $Y \in \sigma \mathcal{S}$ and suppose that $\emptyset \neq \mathcal{T} \subset \mathcal{S}$ with $\mathcal{T} \cup \{Y\}$ having f.i.p.. For each $S \in \mathcal{T}$, (2) implies that $Y \cap S \in \sigma \mathcal{S} \wedge \mathcal{S} \subset \sigma \mathcal{S}$. Since $Y \cap (Y \cap S) = Y \cap S$, (4) implies that $Y \cap S \in \sigma(\{Y\} \cap \mathcal{S})$ for each $\mathcal{S} \in \mathcal{T}$; i.e., $\{Y\} \wedge \mathcal{T} \subset \sigma(\{Y\} \wedge \mathcal{S})$. Moreover, $\{Y\} \wedge \mathcal{T} \subset \gamma(\{Y\} \wedge \mathcal{S})$. Since $\mathcal{T} \cup \{Y\}$ has f.i.p., $\{Y\} \wedge \mathcal{T}$ is not empty and has f.i.p.. Thus (3) implies that $\bigcap(\{Y\} \wedge \mathcal{T}) \neq \emptyset$. It follows that $Y \in \rho \mathcal{S}$. Therefore we have that $\sigma \mathcal{S} \subset \rho \mathcal{S}$ for all $\mathcal{S}$ such that $\gamma \mathcal{S} = \mathcal{S}$. For arbitrary $\mathcal{S}$, (1) implies that $\sigma \mathcal{S} = \sigma(\gamma \mathcal{S}) \subset \rho(\gamma \mathcal{S}) = \rho \mathcal{S}$. Hence $\sigma \leq \rho$.

THEOREM 1. Let $\sigma : 2^{2^X} \to 2^{2^X}$ be an operator. Then $\sigma = \rho$ if and only if σ satisfies the following properties: for each $\mathcal{S} \in 2^{2^X}$,

(1) $\sigma\gamma = \sigma$

(2) $\gamma \mathcal{S} \wedge \sigma \mathcal{S} \subset \sigma \mathcal{S}$

(3) If $\emptyset \neq \mathcal{T} \subset \gamma \mathcal{S} \cap \sigma \mathcal{S}$ and $\mathcal{T}$ has f.i.p., then $\bigcap \mathcal{T} \neq \emptyset$.

(4) For each $A \subset X$, $\{Y \mid Y \cap A \in \sigma\mathcal{S}\} \subset \sigma(\{A\} \wedge \mathcal{S})$.

(5) If $\lambda : 2^{2^X} \to 2^{2^X}$ is an operator which also satisfies (1), (2), (3) and (4), then $\lambda \leq \sigma$.

The proof is an immediate consequence of Proposition 1 and Lemma 1.

REFERENCES.

1. de Groot, J., H.Herrlich, G.E. Strecker, and E.Wattel, Compactness as an operator, submitted for publication.

2. Wattel, E., The Compactness Operator in Set Theory and Topology, Mathematical Centre Tract, 1968.